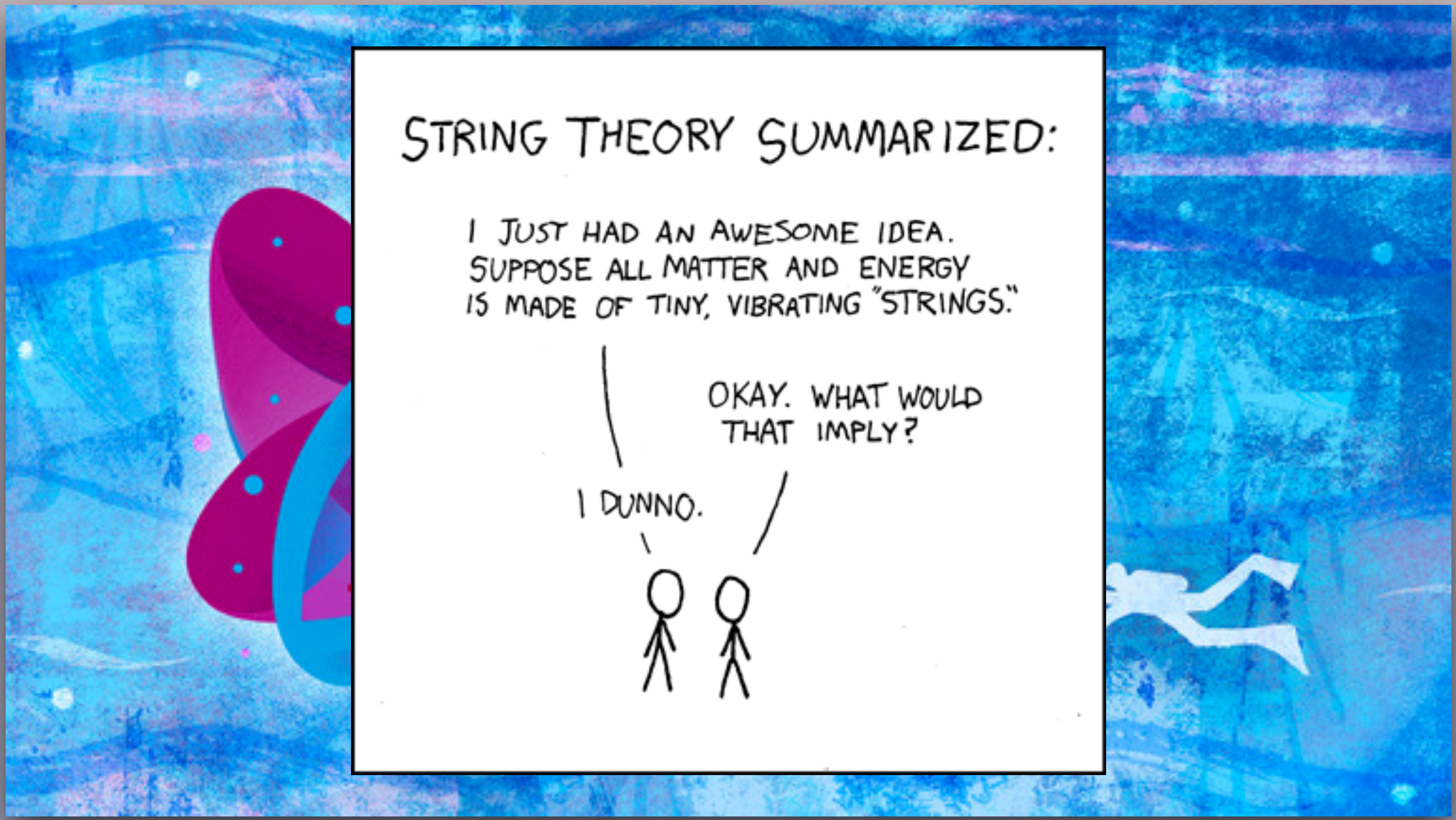
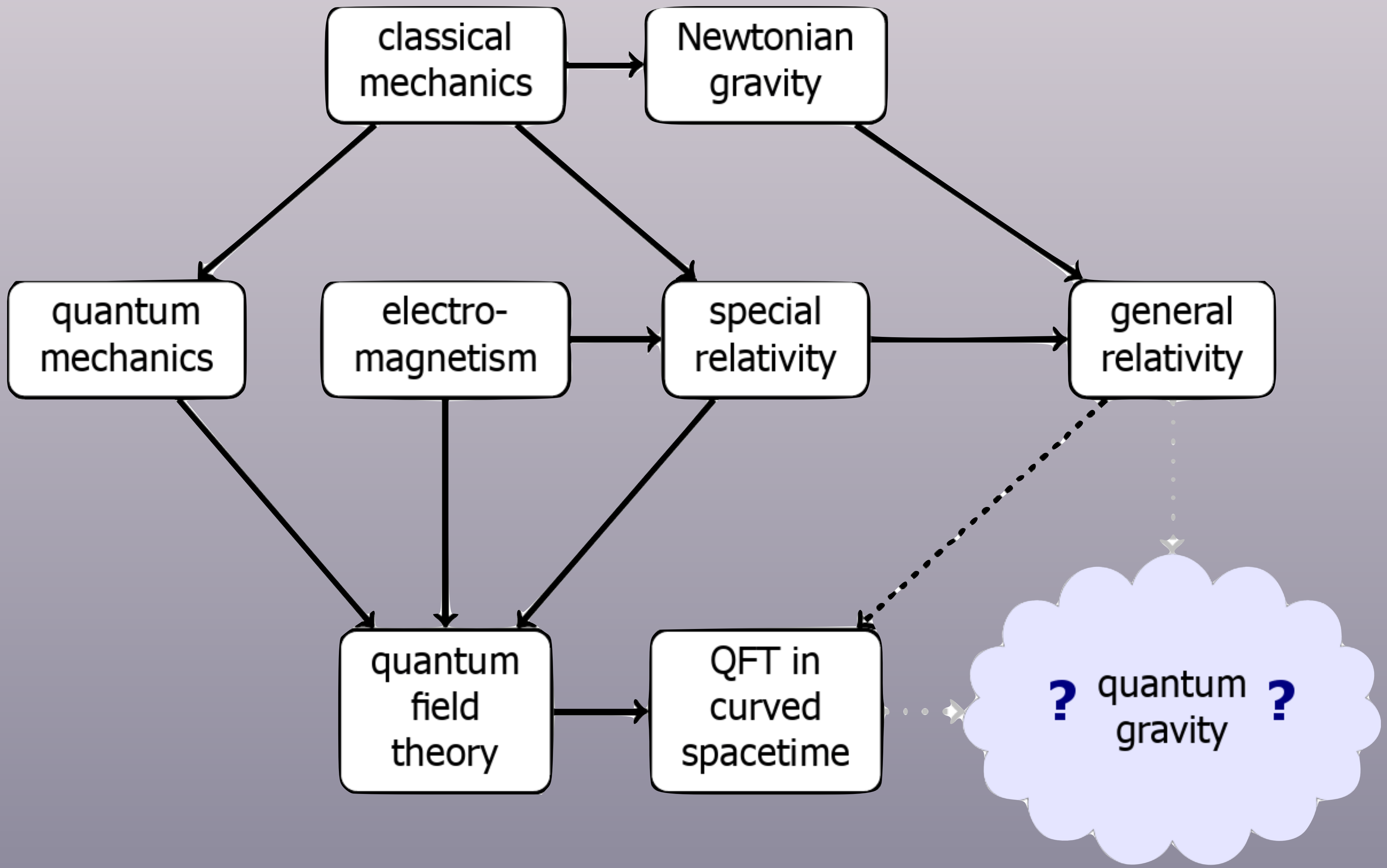
A Theorem About Counting Points on Calabi-Yau Manifolds



A Theorem About Counting Points on Calabi-Yau Manifolds

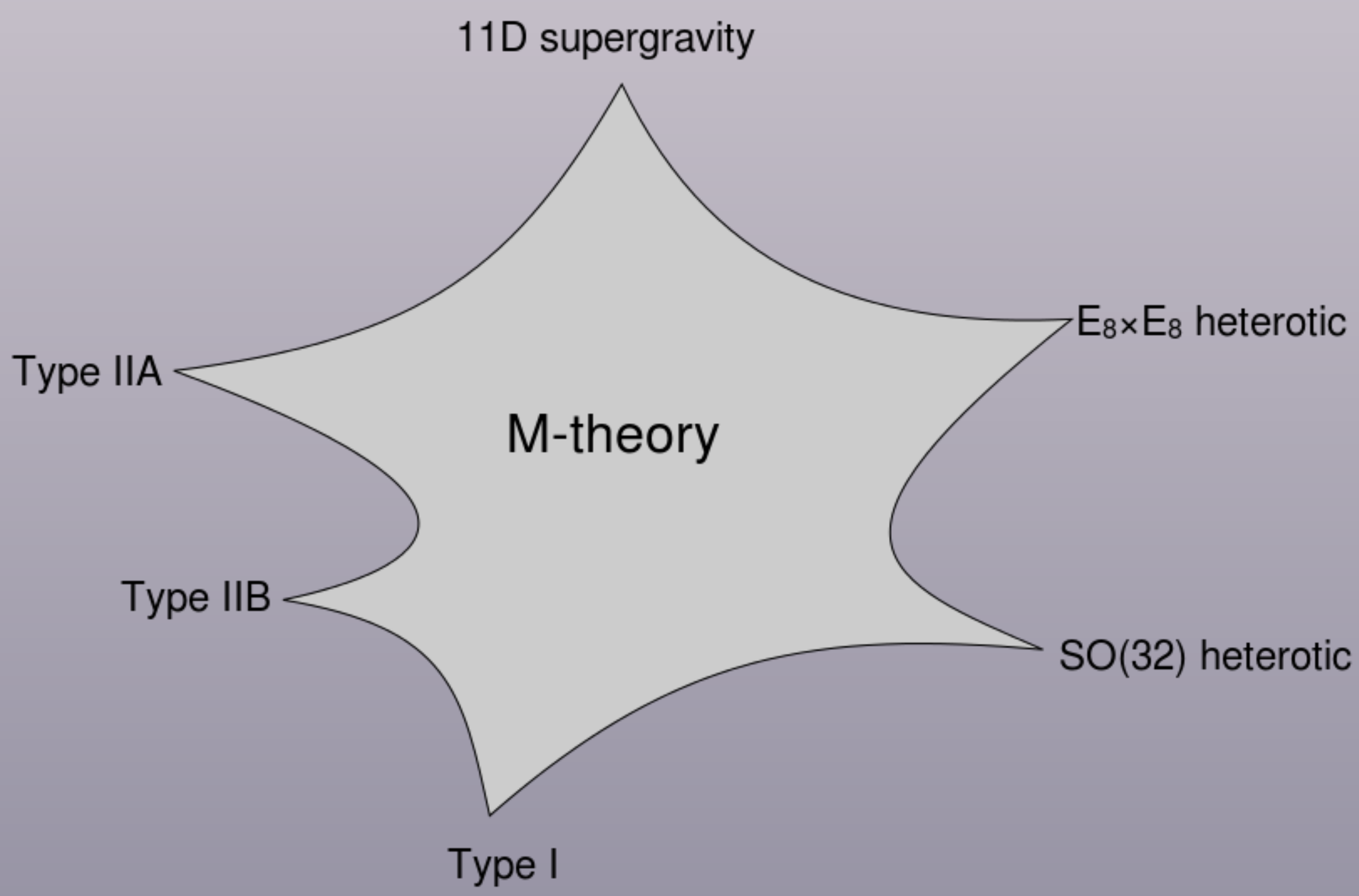
String Theory Fundamentals

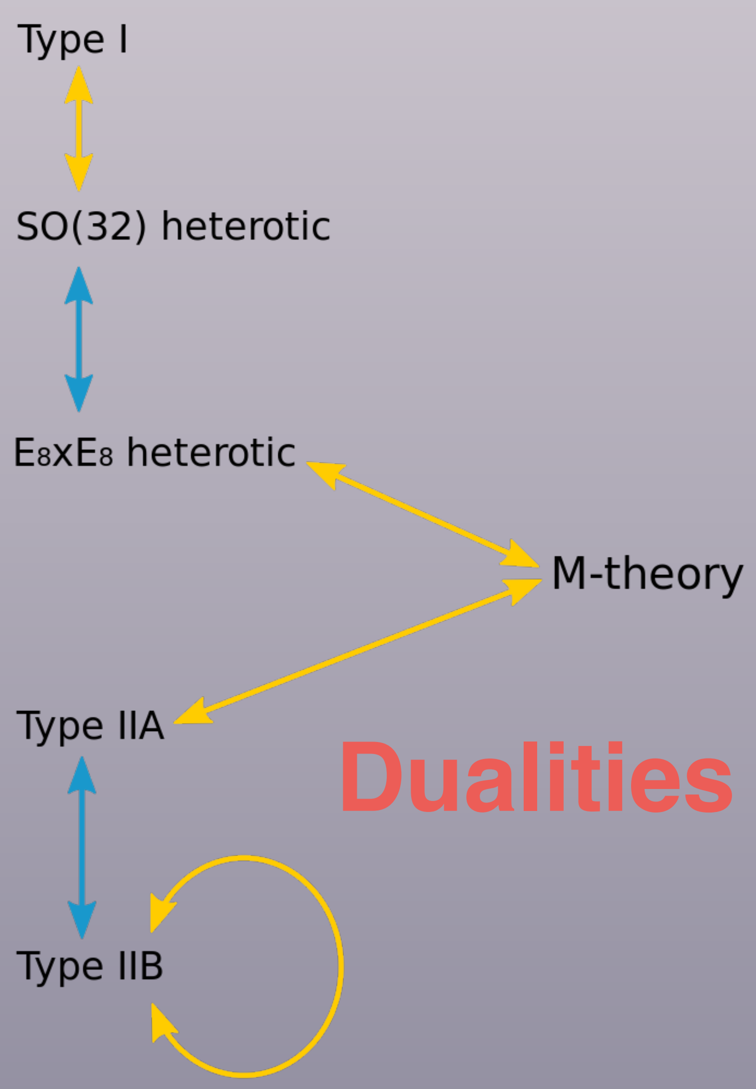


Different String Theories

1994: Second Superstring Revolution

(Witten: different string theories are limits of M-theory)



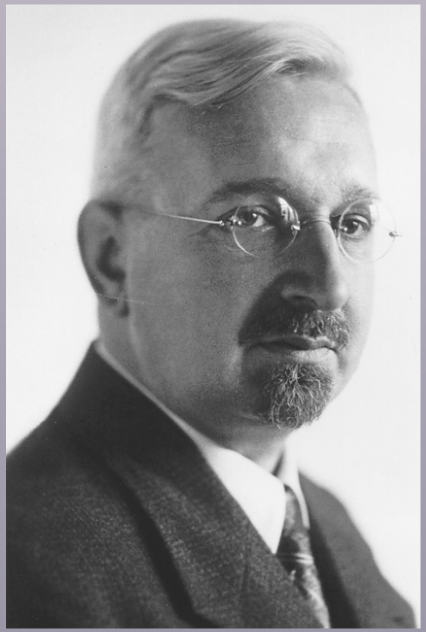


String theories require 10/11 dim space-time.

Extra Dimensions

Einstein (1915): 4dim space-time
(gravity is caused by space-time curvature)

Kaluza-Klein (1919): 4dim space-time + 1extra dim
(electro-magnetism is caused by curvature in extra dim)



Theodor Kaluza

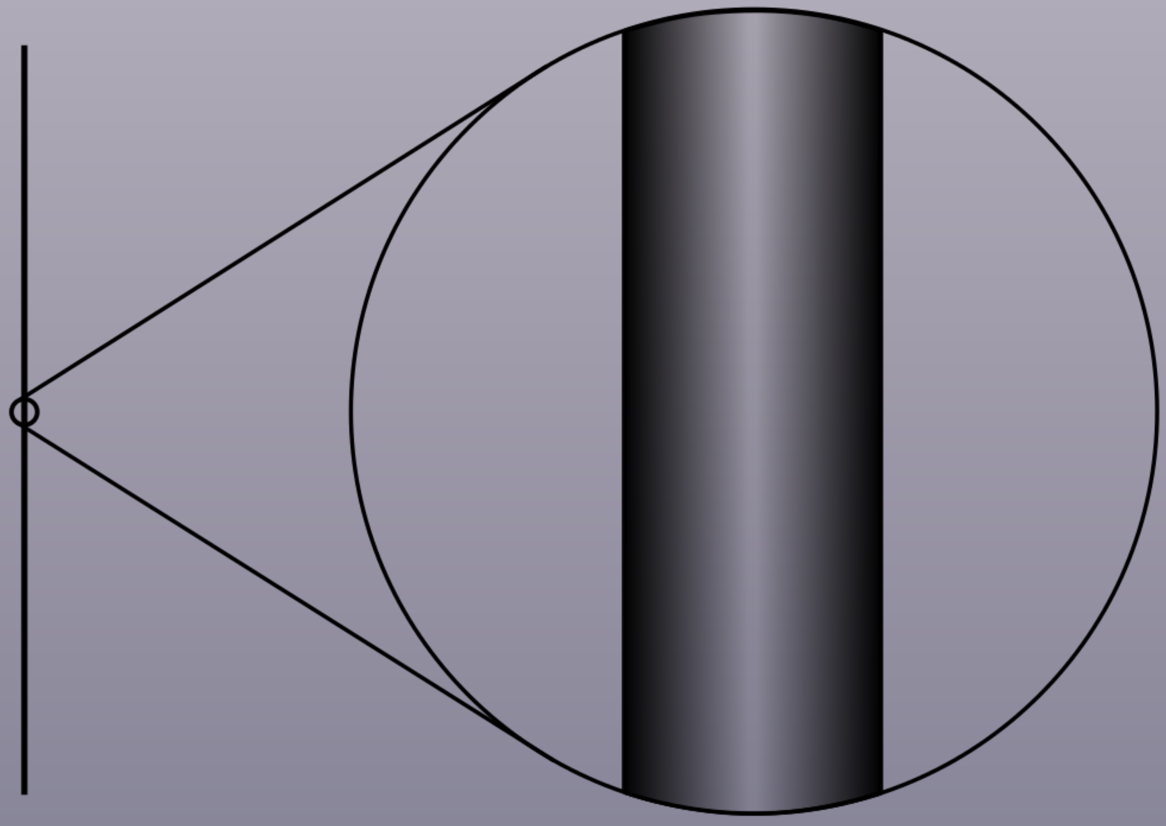
Oscar Klein

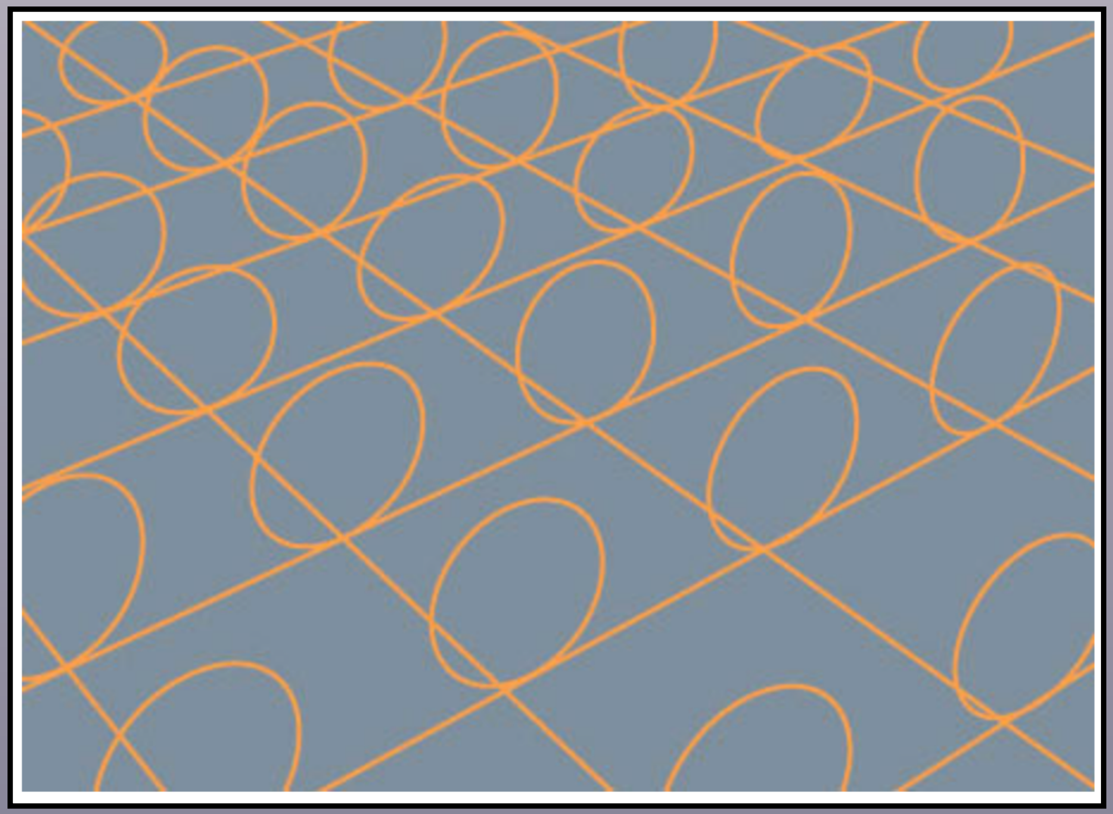
Extra Dimensions

Einstein (1915): 4dim space-time
(gravity is caused by space-time curvature)

Kaluza-Klein (1919): 4dim space-time + 1extra dim
(electro-magnetism is caused by curvature in extra dim)

Make extra dimension compact and hide it.





additional spatial dimension curled up within every point

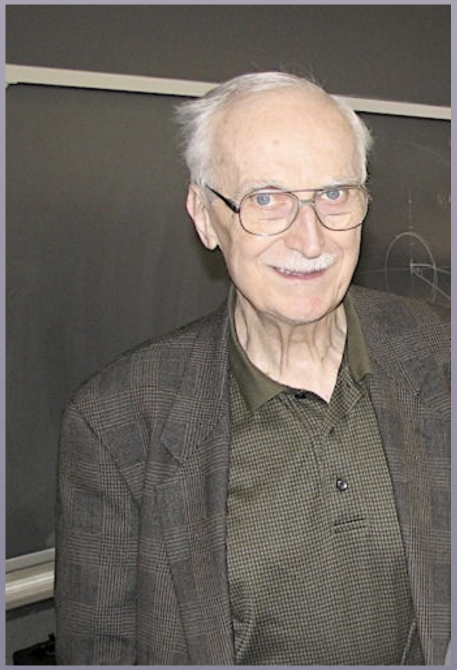
Extra Dimensions

Superstring theories live in **10** or 11 dimensions!

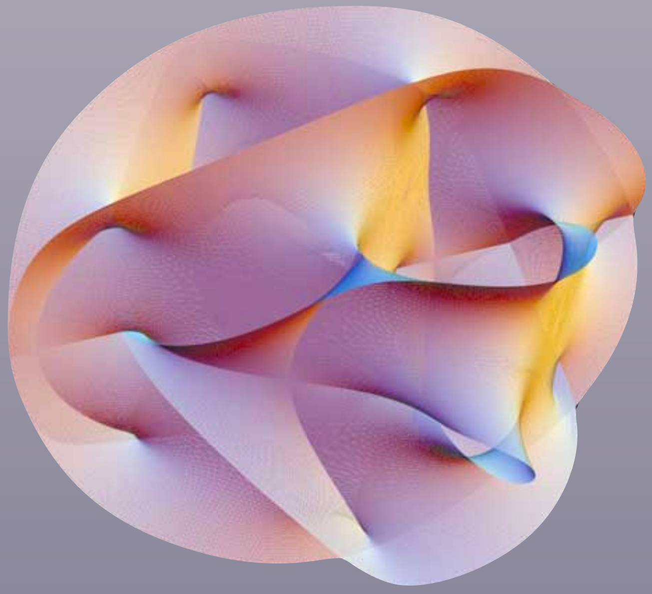
We need a **compact 6-dimensional space**,
to make an effectively 4-dimensional theory.

Compact space has to have special properties
to produce a theory that can describe nature.

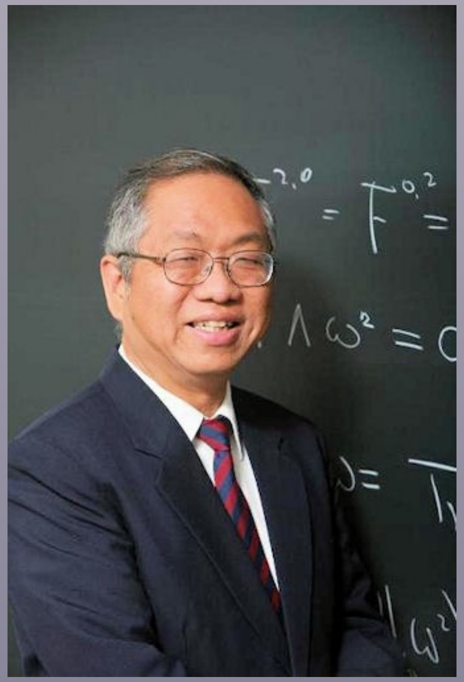
Extra dimensions must be shaped as a
Calabi-Yau manifold.



Eugenio Calabi



Quintic CY



Shing-Tung Yau

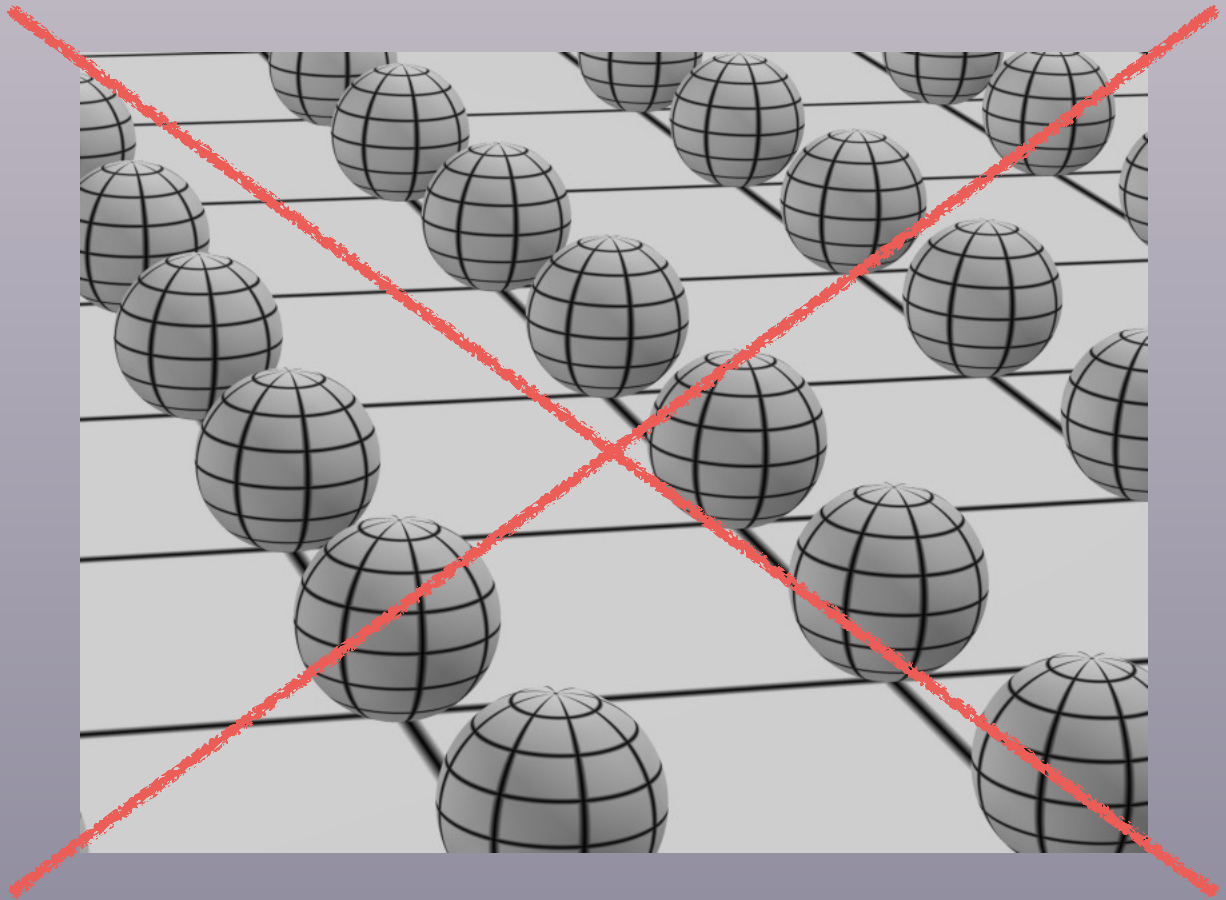
Extra Dimensions

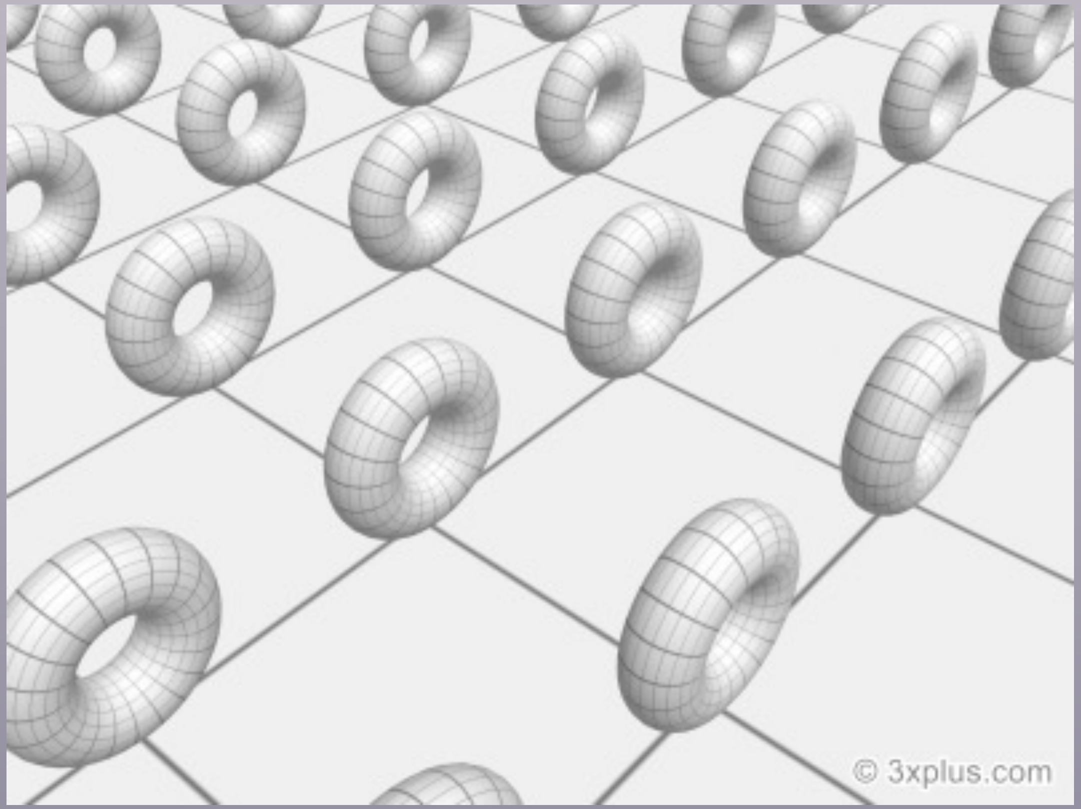
Extra dimensions must be shaped as a

Calabi-Yau manifold.

(compact, complex, 2n-dimensional Kähler-manifold with vanishing Ricci curvature)

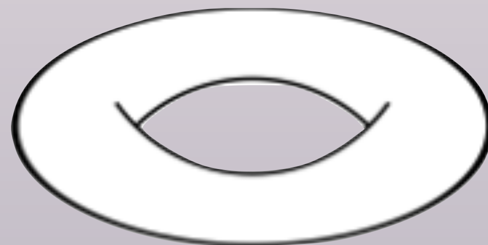
2-dimensional example:





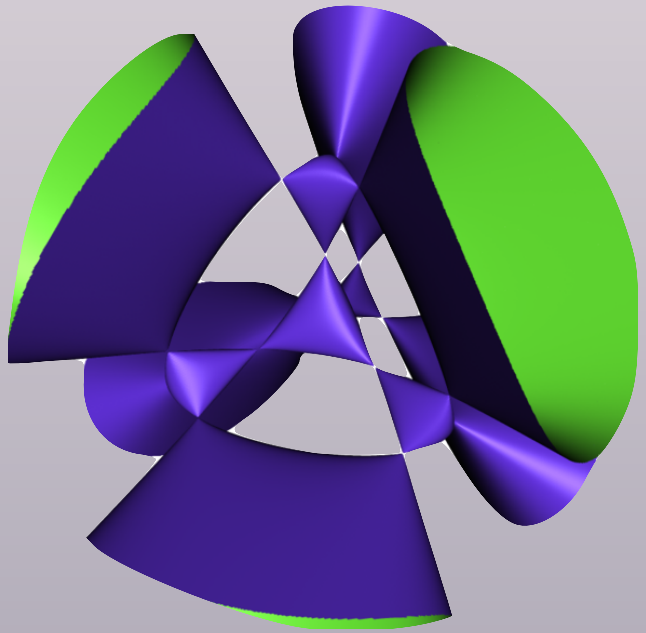
Calabi-Yau Manifolds

- **2dim**: torus



$$\chi = 0$$

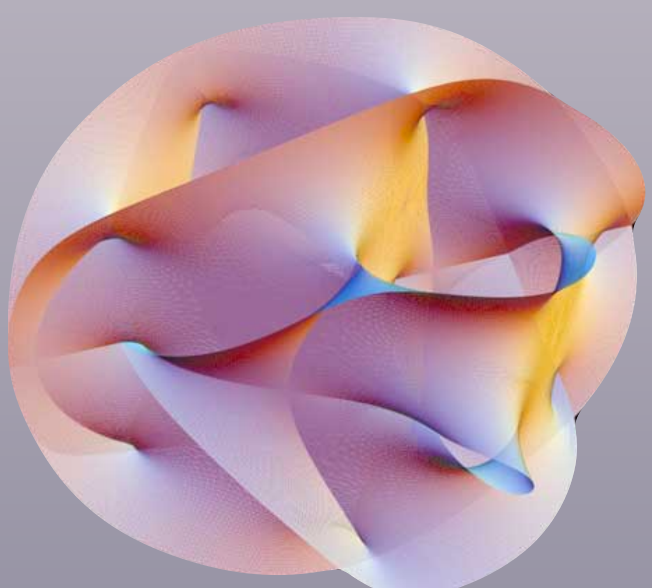
- **4dim**: K3 surface (simply connected)



Kummer Surface

$$\chi = 24$$

- **6dim**: many examples, no general structure known



Quintic CY

$$\chi = -200$$

Dualities in string theory

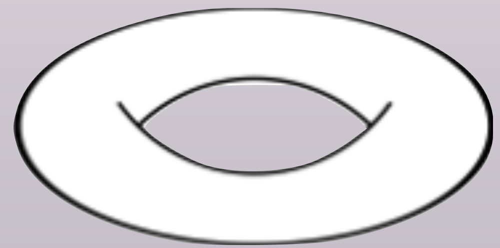
Mirror symmetry for CYs:

For every CY with χ,

there is a CY with $-\chi$.

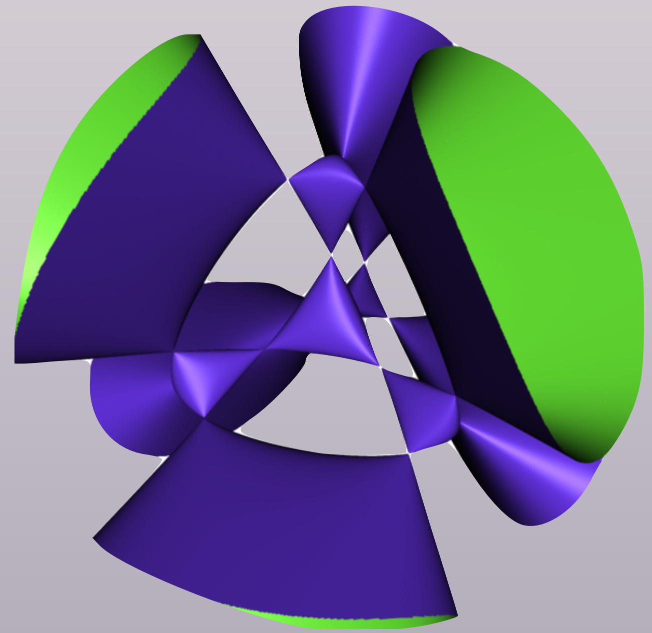
Calabi-Yau Manifolds

- **2dim**: torus



$\chi = 0$

- **4dim**: K3 surface (simply connected)



Kummer Surface

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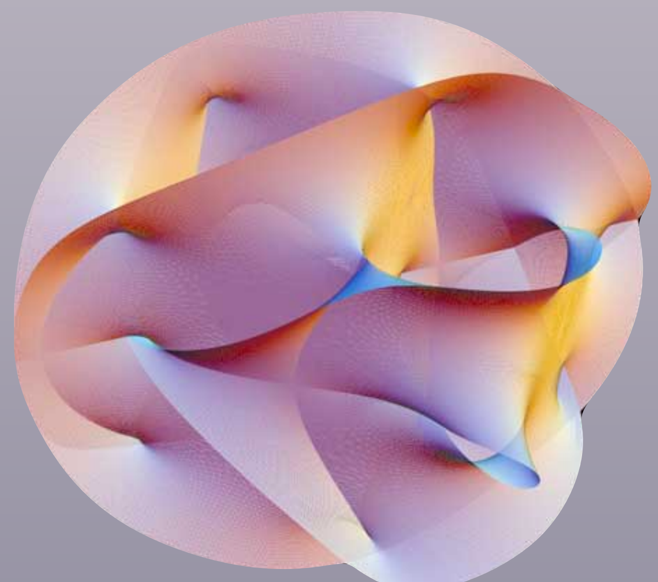


Quintic CY

$\chi = -200$

Dualities in string theory

Mirror symmetry for CYs:
For every CY with χ, there is a CY with $-\chi$.

Mirror Symmetry for K3 Surfaces

General Smooth K3
(symplectic manifold)
(A-Side)

Family of Complex K3 Manifolds
(B-Side)

Greene-Plesser
(divide by symmetry
group and resolve
singularities)

Quartic in $\mathbb{P}^3$
(singular K3)

$$x^4 + y^4 + w^4 + z^4 + 4\lambda xywz = 0$$

Dwork Pencil

<u>What makes these K3 surfaces mirrors?</u>

- Equivalent Hodge Structures
- “Same” Rational Point Counts

Counting Rational Points on Elliptic Curves (Complex One-Dimensional Cubics in $\mathbb{P}^2$)

Family of Elliptic Curves (tori):

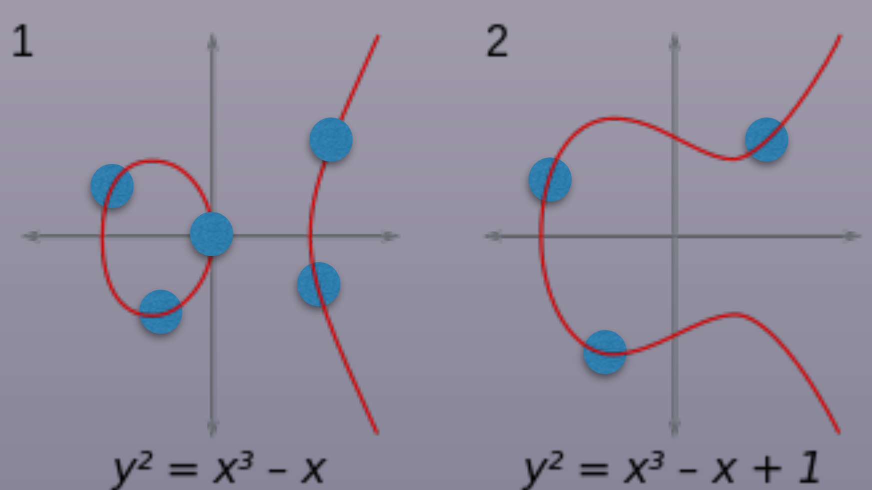
$$-y^2 z + x(x-z)(x-\lambda z) = 0$$

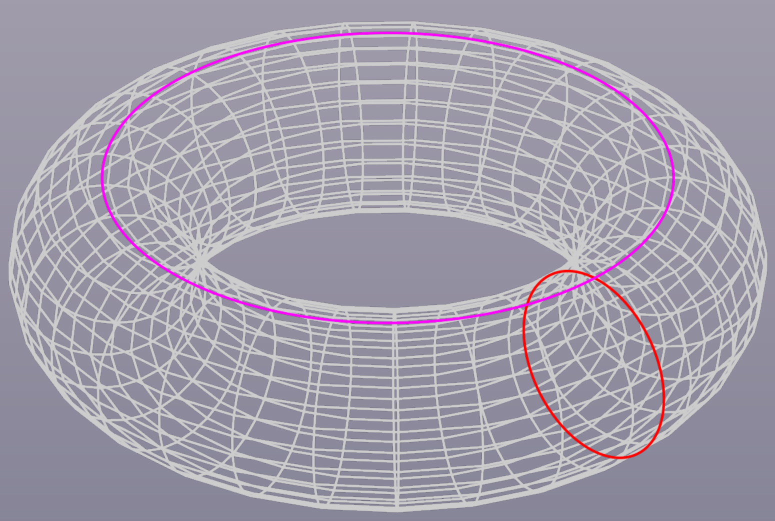
Elliptic Integrals:

$$\int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-\lambda)}} = {}_2F_1(\frac{1}{2}, \frac{1}{2}, 1, ; \lambda) = 1 + \frac{1}{4}\lambda + \frac{9}{64}\lambda^2 + \frac{25}{256}\lambda^3 + \ldots$$

Counting Function for Family of Elliptic Curves:

$$|X_\lambda| = -(-1)^{\frac{p-1}{2}} {}_2F_1\left(\frac{1-p}{2}, \frac{1-p}{2}, 1; \lambda\right) \text{ mod p}$$





Generalizing the Dwork Pencil

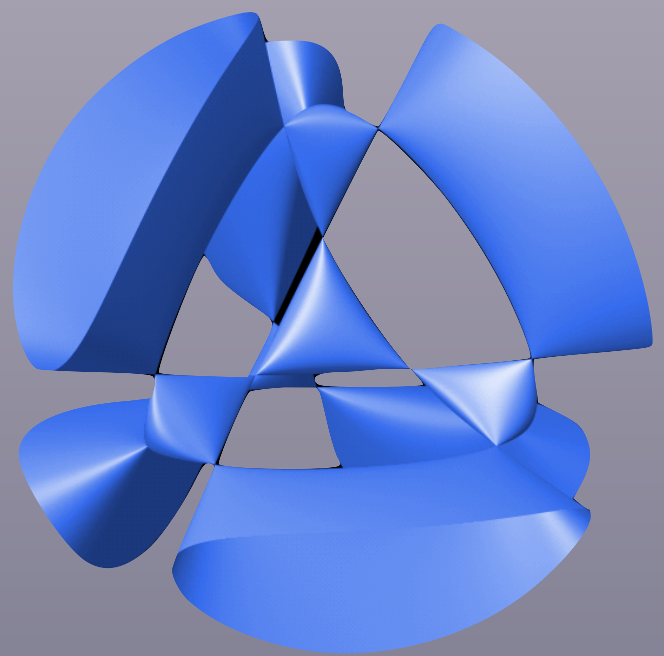
The one-parameter Dwork Pencil

$$x^4 + y^4 + w^4 + z^4 + 4\lambda xywz = 0$$

The three-parameter Kummer quartic serves as the natural generalization of the Dwork pencil, as it preserves some of the symmetry.

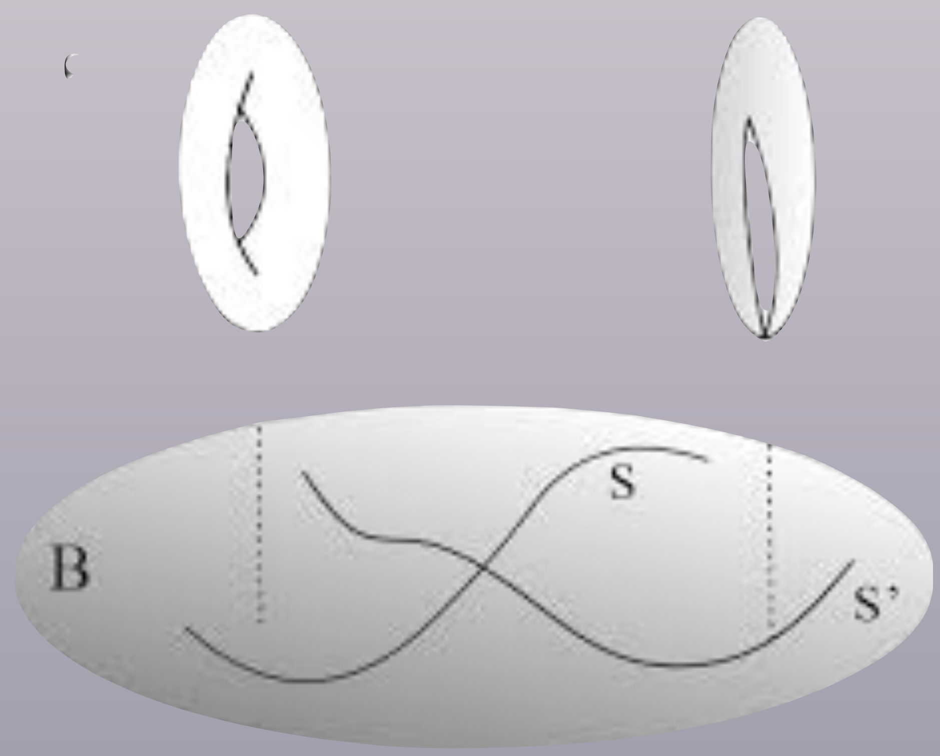
$$x^4 + y^4 + w^4 + z^4 + 2Dxywz - A(x^2z^2 + y^2w^2) - B(x^2y^2 + w^2z^2) - C(x^2w^2 + y^2z^2) = 0$$

$A, B, C, D \in \mathbb{C}$ such that $D^2 = A^2 + B^2 + C^2 + ABC - 4$



Generalizing the Greene-Plesser Mechanism

We can establish elliptic fibrations both on our three-parameter Kummer quartic and the mirror of the Dwork pencil.



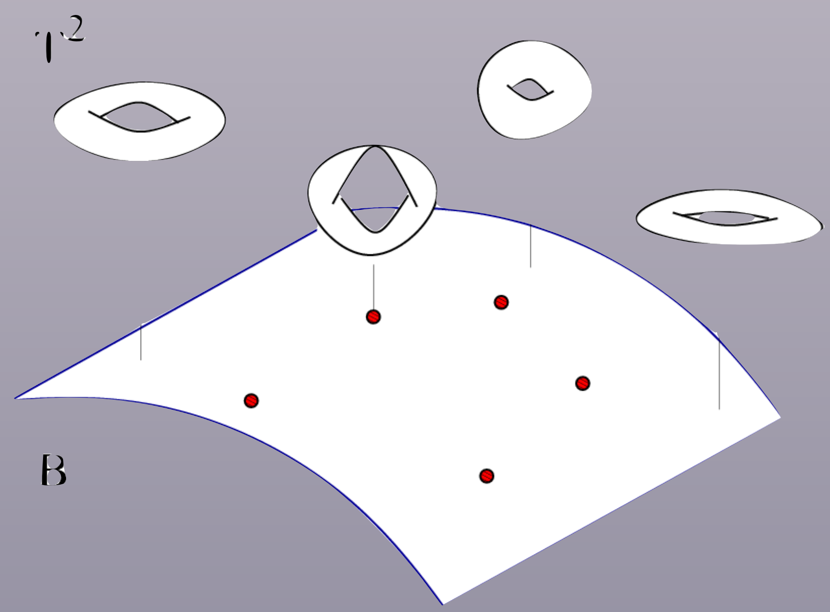
The elliptic fibration structure allows us to generalize the Greene-Plesser mechanism. We then obtain the three-parameter generalization of the mirror of the generalized Dwork pencil.

Counting Rational Points on K3 Surfaces

We established a three-parameter family of K3 surfaces generalizing the Greene-Plesser mechanism.

$$x^4 + y^4 + w^4 + z^4 + 2Dxywz - A(x^2z^2 + y^2w^2) - B(x^2y^2 + w^2z^2) - C(x^2w^2 + y^2z^2) = 0$$

Through using the elliptic fibration on the three-parameter mirror family, we prove the following theorem:



Theorem: The counting function (of rational points) on the three-parameter family of generalized mirror K3 surfaces can be computed explicitly (it is a multivariate generalization of the Gauss hypergeometric function).

Thank You!

Acknowledgements: Dr. Malmendier, Dr. Sung, USU URCO Grant Program, and USU Mini Grant Program.