Kummer Surfaces

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What is a Kummer Surface?

- A Kummer surface is the minimum resolution of the quotient of an abelian surface over the complex numbers and the minus identity involution.

$$Kum(A) = A/ < -\mathbb{I} >$$

- We will consider the case where A is the Jacobian of a genus two curve.

$$Kum(Jac(C)) = Jac(C)/ < -\mathbb{I} >$$

Genus Two Curves and the Shioda Sextic

- In weighted projective space, the Rosenhain normal form of a genus two curve is given as

$$C : Y^2 = XZ(X - Z)(X - \lambda_1 Z)(X - \lambda_2 Z)(X - \lambda_3 Z)$$

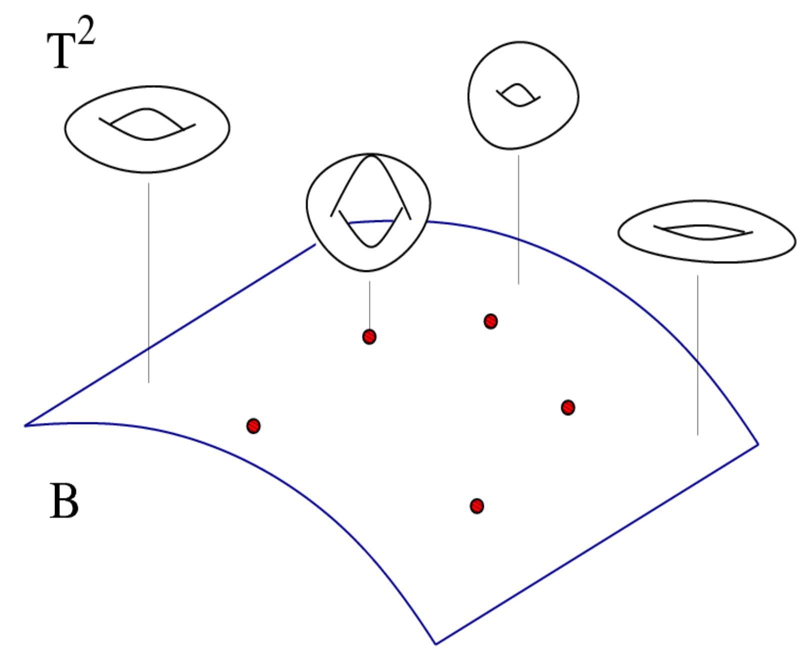
- The Shioda sextic is the hypersurface obtained through taking the quotient of $C^{(2)}$ with its hyper elliptic product involution.

$$\tilde{z_4}^2 = z_1 z_3 (z_1 - z_2 + z_3) \prod_{i=1}^{3} (\lambda_i^2 z_1 - \lambda_i z_2 + z_3)$$

- For a genus two curve in Rosenhain normal form, the Shioda sextic is birational to $Kum(Jac(C))$

Kummer Surfaces and Elliptic Fibrations

- We can analyze the Shioda sextic as an elliptic fibration over the Riemann sphere through bringing it into Weierstrass normal form.



$$Y^2 = 4X^3 - g_2(T)X - g_3(T)$$

$$\mathbb{P}^1 = \left\{T \in \mathbb{C}\right\} \cup \left\{\infty\right\}$$

- This fibration emits the following singular fibers and Mordell-Weil group

$$2I_0^* + 6I_2 \qquad MW = (\mathbb{Z}/2\mathbb{Z})^2 \oplus < 1 >$$

Normal Forms for Kummer Surfaces

- The Shioda sextic defines a double cover of two-dimensional projective space branched along six lines which are tangent to a common conic.

$$z_1 = 0,\ z_3 = 0,\ z_1 - z_2 + z_3 = 0,\ \lambda_i^2 z_1 - \lambda_i z_2 + z_3 = 0, \text{ where } 1 \leq i \leq 3$$

$$z_2^2 - 4z_1 z_3 = 0$$

- We can define a proper transformation from the Shioda sextic to a nodal quartic in three-dimensional projective space known as the Cassels-Flynn quartic.

$$K_2(z_1, z_2, z_3)z_4^2 + K_1(z_1, z_2, z_3)z_4 + K_0(z_1, z_2, z_3) = 0$$

Normal Forms for Kummer Surfaces

- The Cassels-Flynn quartic is isomorphic to the Gopel-Hudson quartic in three-dimensional projective space.

$$0 = X_0^4 + X_1^4 + X_2^4 + X_3^4 + 2DX_0X_1X_2X_3 - A(X_0^2X_3^3 + X_1^2X_2^2) - B(X_0^2X_1^2 + X_2^2X_3^2) - C(X_0^2X_2^2 + X_1^2X_3^2)$$

$$A, B, C, D \in \mathbb{C} \text{ such that } D^2 = A^2 + B^2 + C^2 + ABC - 4$$

- In letting $A = B = C = 0 \text{ and } D = 2\lambda \text{ for } \lambda \in \mathbb{C}$, we obtain a one-parameter family of quartics in three-dimensional projective space, and, once we have resolved the singularities, a one-parameter family of K3 surfaces.

Greene-Plesser Orbifolding and Mirror Symmetry

- Through the Greene-Plesser orbifolding procedure, we are allowed to construct the mirror family of K3 surfaces for our special family of K3 surfaces.

$$x_1 x_2 x_3 (x_1 + x_2 + x_3 + 1) + \frac{t}{4^4} = 0$$

$$x_1 = \frac{X_1^3}{4X_0 X_2 X_3 \lambda},\ x_2 = \frac{X_2^3}{4X_0 X_1 X_3 \lambda},\ x_3 = \frac{X_3^3}{4X_0 X_1 X_2 \lambda},\ t = \frac{1}{\lambda^4}$$

Mirror Family and Elliptic Fibrations

- We can analyze the mirror family of K3 surfaces as an elliptic fibration through bringing it into Weierstrass normal form.

$$Y^2 = 4X^3 - g_2\Big(-\frac{3^3t}{4^4u^3(u+1)}\Big)(u(u+1))^4 - g_3\Big(-\frac{3^3t}{4^4u^3(u+1)}\Big)(u(u+1))^6$$

- This fibration emits the following singular fibers and Mordell-Weil group

$$IV^* + I_{12} + 4I_1 \qquad\qquad MW = (\mathbb{Z}/3\mathbb{Z})^2$$