Isogenies of K3 Surfaces of Rank 18

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Background

- A correspondence from a variety X to a variety Y is taken to be a subset Z of $X \times Y$ such that Z is finite and surjective over each component of X.
- In (van Geemen, Top, 2006), a dominant rational map from the one-parameter twisted Legendre pencil $\mathcal{X}_t$ to $Kum(\mathcal{E}_t \times \mathcal{E}_t^{(t+1)})$ was constructed over a finite extension of Q(t).
- In (Ahlgren, Ono, Penniston, 2002), certain Galois representations related to the K3 surfaces and abelian surfaces involved were shown to be isomorphic. This suggested the existence of an algebraic cycle realizing correspondence between $\mathcal{X}_t$ and $Kum(\mathcal{E}_t \times \mathcal{E}_t^{(t+1)})$. van Geemen and Top realized this algebraic cycle explicitly as the graph of the rational map in the cartesian product.

Results

- We compute explicit geometric isogenies between $Kum(Jac(\mathcal{C}_0))$, $Kum(\mathcal{E}_1 \times \mathcal{E}_2)$, and $\mathcal{X}$. Additionally, we compute normal forms for these K3 surfaces as quartic hyper surfaces, double quadric surfaces, double sextic surfaces, and Jacobian elliptic surfaces.
- The double coverings define correspondences between $Kum(Jac(\mathcal{C}_0))$ and $\mathcal{X}$, and $\mathcal{X}$ and $Kum(\mathcal{E}_1 \times \mathcal{E}_2)$, generalizing the results of van Geemen and Top.

Overview

1. Gluing Construction
2. Abelian Surfaces
3. K3 Surfaces
4. Legendre Pencil

Gluing Construction

We consider a genus two curve with elliptic involution $\mathcal{C}_0$

The composition of the elliptic involution with the hyperelliptic involution defines a second elliptic involution. Both elliptic involutions define rational quotient maps.

$$\mathcal{E}_1 \xleftarrow{\pi_{\mathcal{E}_1}} \mathcal{C}_0 \xrightarrow{\pi_{\mathcal{E}_2}} \mathcal{E}_2$$

For the six Weierstrass points of $\mathcal{C}_0$ and the two ramification points of both double covers:

$$\left.\begin{matrix} \{P_2, P_3\} \\ \{P_1, P_5\} \\ \{P_5, P_6\} \\ \{Q_{2,+}, Q_{2,-}\} \end{matrix}\right\} \xrightarrow{\pi_{\mathcal{E}_1}} \mathcal{E}_1[2]$$

$$\left.\begin{matrix} \{P_2, P_3\} \\ \{P_1, P_5\} \\ \{P_5, P_6\} \\ \{Q_{1,+}, Q_{1,-}\} \end{matrix}\right\} \xrightarrow{\pi_{\mathcal{E}_2}} \mathcal{E}_2[2]$$

Gluing Construction

We associate with the branch locus of $\pi_{\mathcal{E}_1}$ and $\pi_{\mathcal{E}_2}$ the effective divisors

$$B_1 = [\pi_{\mathcal{E}_1}(Q_{1,+}) + \pi_{\mathcal{E}_1}(Q_{1,-})] \quad B_2 = [\pi_{\mathcal{E}_2}(Q_{2,+}) + \pi_{\mathcal{E}_2}(Q_{2,-})]$$

Conversely, for $\mathcal{L}_l^{\otimes 2} \cong \mathcal{O}_{\mathcal{E}_l}(B_l)$, the data $(\mathcal{E}_l, \mathcal{L}_l, B_l)$ determines $\mathcal{C}_0$ uniquely, and is equivalent to the unordered pair $\{\Lambda_1, \Lambda_2\}$, where $\Lambda_1 \neq \Lambda_2$ and

$$\mathcal{C}_0 \cong \left\{[X:Y:Z] | Y^2 = \left(X^2 - Z^2\right)\left(X^2 - \frac{\Lambda_1}{\Lambda_2}Z^2\right)\left(X^2 - \frac{\Lambda - 1}{\Lambda_2 - 1}Z^2\right)\right\}$$

Abelian Surfaces

We proceed using the results of (Clingher, Malmendier, Shaska, 2019) and (Clingher, Malmendier, Shaska, 2020).

We consider the principally polarized abelian surface $Jac(\mathcal{C}_0)$ and its group of 2-torsion points.

$$A = Jac(\mathcal{C}_0)$$

$$P_{ij} = [P_i + P_j - 2P_6] \in A[2], \text{ with } 1 \leq i < j \leq 6$$

There are 15 inequivalent Goepel groups.

$$\{P_{66}, P_{ij}, P_{kl}, P_{mn}\} \text{ such that } \{i, j, k, l, m, n\} = \{1, \ldots, 6\}$$

<u>Proposition</u>: For our 15 Goepel groups,

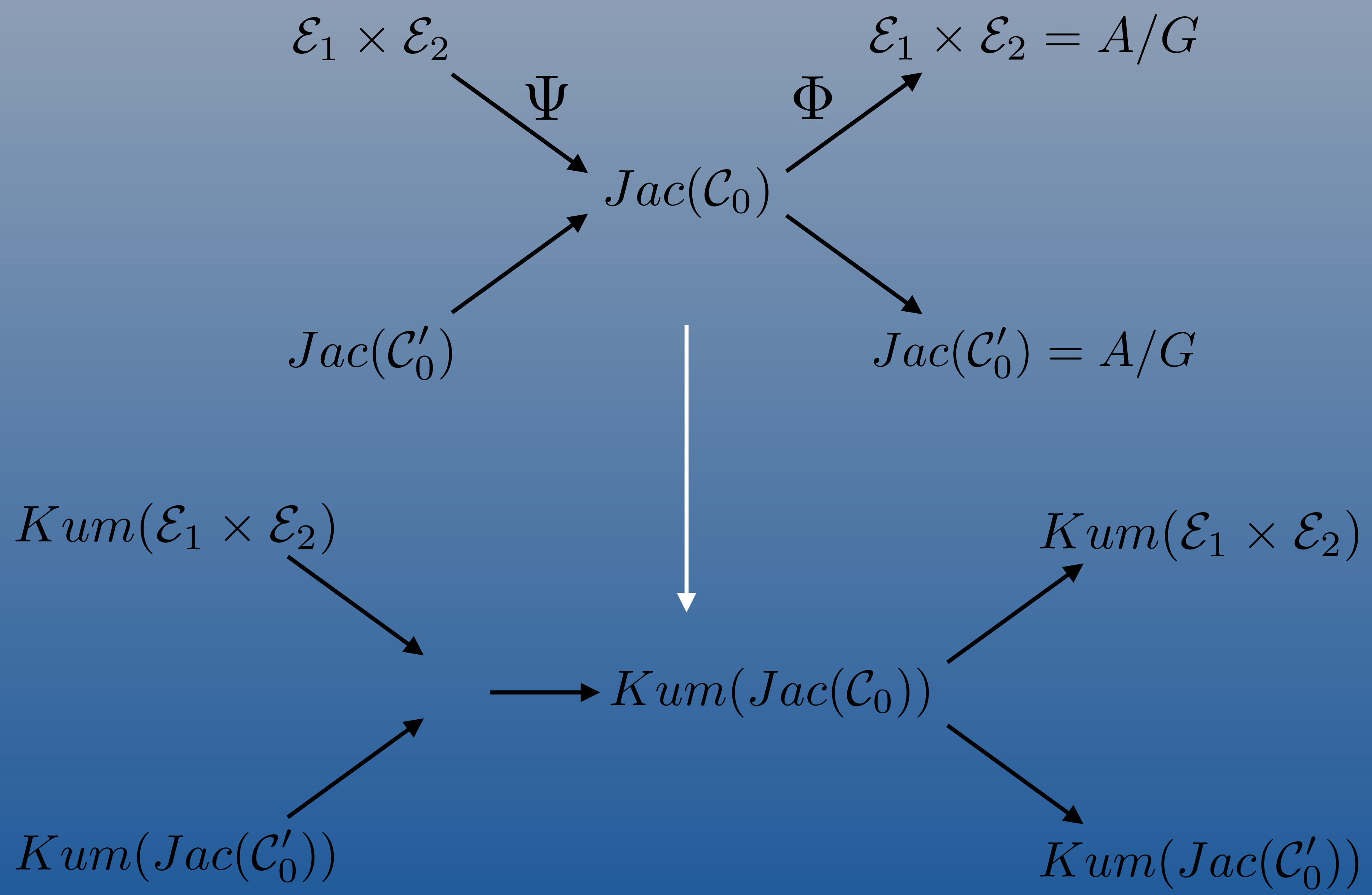
- One is such that $A/G \cong \mathcal{E}_1 \times \mathcal{E}_2$ (and $T = H \oplus H$)
- Six are such that $A/G' \cong Jac(\mathcal{C}_0') \in \mathcal{H}_4$ (and $T = H \oplus \langle 2 \rangle \oplus \langle -2 \rangle$)
- Eight are such that $A/G'' \cong Jac(\mathcal{C}_0') \in \mathcal{H}_{16}$ (and $T = H \oplus \langle 4 \rangle \oplus \langle -4 \rangle$)

The 15 components of the Humbert surface $\mathcal{H}_4$ in $\mathbb{H}/\Gamma_2(2)$ can be identified with the 15 Goepel groups.

K3 Surfaces

The different sets of Goepel groups allow us to define (2,2)-isogenies, where

$\Psi(P,Q) = [\pi_{\mathcal{E}_1}(P) + \pi_{\mathcal{E}_2}(Q) - 2P_{66}]$



The image of two-torsion points label even eights on a Kummer surface. Taking the double cover branched along these even eights gives us new K3 surfaces. We then quotient by a Nikulin involution produced by translation by a 2-torsion point.

K3 Surfaces

The Kummer surface $Kum(Jac(\mathcal{C}))$ can be regarded as the minimal resolution of a quartic hypersurface in three-dimensional projective space.

$$W^4 + X^4 + Y^4 + Z^4 + 2DWXYZ - A(W^2Z^2 + X^2Y^2) - B(W^2X^2 + Y^2Z^2) - C(W^2Y^2 + X^2Z^2) = 0$$

<u>Theorem</u>: Over $\mathbb{Q}(\Lambda_1, \Lambda_2)$, we have a similar quartic for $Kum(Jac(\mathcal{C}_0))$ if and only if one of 15 relations hold.

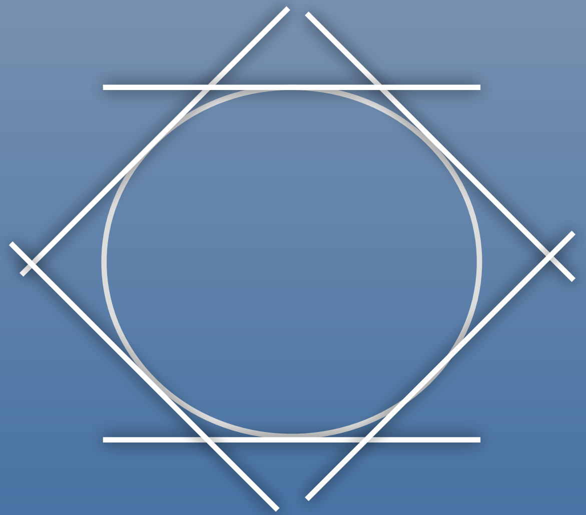
$$Z_{00}^4 + (1-\Lambda_1)(1-\Lambda-2)Z_{01}^4 + \Lambda_1\Lambda_2 Z_{10}^4 + \Lambda_1\Lambda_2(1-\Lambda_1)(1-\Lambda_2)Z_{11}^4 - (2-\Lambda_1-\Lambda_2)(Z_{00}^2 Z_{01}^2 + \Lambda_1\Lambda_2 Z_{10}^2 Z_{11}^2)$$

$$-(2\Lambda_1\Lambda_2 - \Lambda_1 - \Lambda_2)(Z_{00}^2 Z_{11}^2 + Z_{01}^2 Z_{10}^2) - (\Lambda_1 + \Lambda_2)(Z_{00}^2 Z_{10}^2 + (1-\Lambda_1)(1-\Lambda_2)Z_{01}^2 Z_{11}^2) = 0$$

The 15 components of $\mathcal{H}_4$ can be identified with these relations.

K3 Surfaces

$Kum(Jac(\mathcal{C}))$ is birationally equivalent to a double sextic surface branched along the union of six lines tangent to a common conic.



$Jac(\mathcal{C}_0) \in \mathcal{H}_4$ if and only if three non-adjacent nodes in the Kummer plane are co-linear. (Birkenhake, Wilhelm, 2003). We can identify the co-linear points in the Kummer plane to the corresponding Goepel group.

Components of $\mathcal{H}_4 \in \mathbb{H}/\Gamma_2(2)$

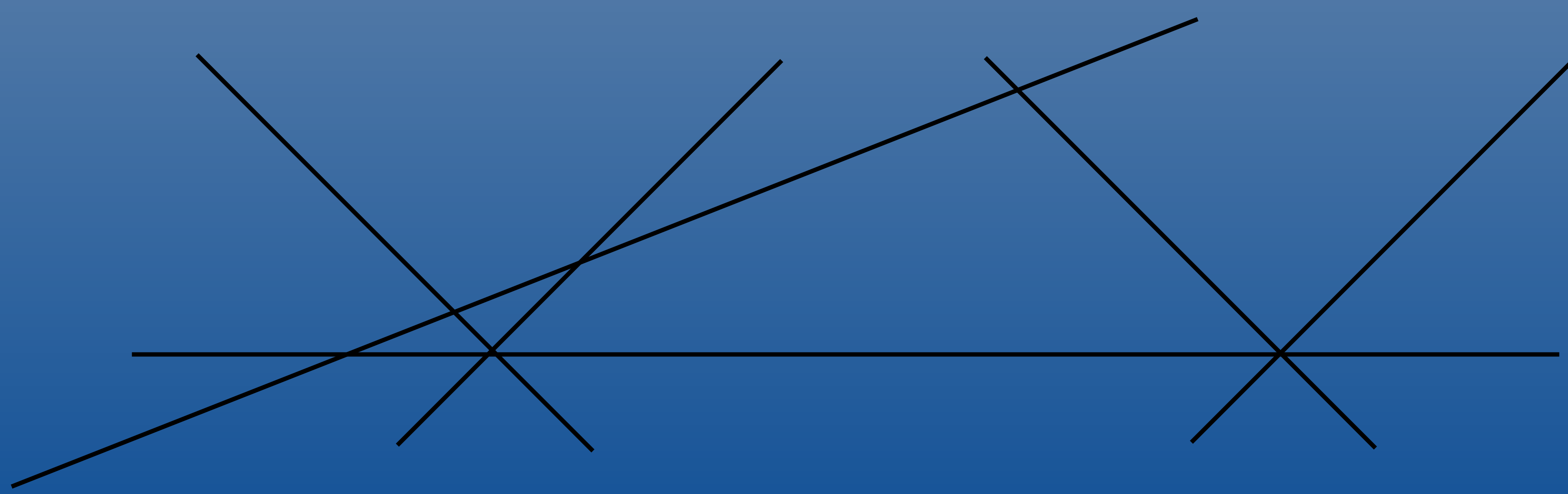
	I		
$2\tau_{12} + \tau_{11}\tau_{22} - \tau_{12}^2 = 0$	$\lambda_1 = \lambda_2\lambda_3$	$D = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{23}, \mathsf{p}_{46}\}$
	II		
$\tau_{11} + 2\tau_{12} = 0$	$\lambda_1 - \lambda_2 = \lambda_3(1 - \lambda_2)$	$B + C = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{23}, \mathsf{p}_{56}\}$
$\tau_{11} + 2\tau_{12} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1(1 - \lambda_3) = \lambda_2(\lambda_1 - \lambda_3)$	$B - C = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{23}, \mathsf{p}_{45}\}$
	III		
$2\tau_{12} - \tau_{22} = 0$	$\lambda_3 = \lambda_1\lambda_2$	$A - C = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{35}, \mathsf{p}_{46}\}$
$2\tau_{12} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_2 = \lambda_1\lambda_3$	$A + C = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{25}, \mathsf{p}_{46}\}$
$\tau_{11} - \tau_{22} = 0$	$\lambda_1 - \lambda_2 = \lambda_2(\lambda_1 - \lambda_3)$	$A + B = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{26}, \mathsf{p}_{34}\}$
$\tau_{11} - \tau_{22} + (\tau_{11}\tau_{22} - \tau_{12}^2) = 0$	$\lambda_1 - \lambda_3 = \lambda_3(\lambda_1 - \lambda_2)$	$A - B = 0$	$\{\mathsf{p}_{15}, \mathsf{p}_{24}, \mathsf{p}_{36}\}$
	IV		
$2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = -\lambda_1(1 - \lambda_2)$	$-A + B - C - D + 6 = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{34}, \mathsf{p}_{56}\}$
$2\tau_{12} + \tau_{22} = 1$	$\lambda_3(1 - \lambda_2) = -\lambda_1(\lambda_2 - \lambda_3)$	$A + B + C + D + 6 = 0$	$\{\mathsf{p}_{12}, \mathsf{p}_{36}, \mathsf{p}_{45}\}$
$\tau_{11} + 2\tau_{12} = 1$	$\lambda_2 - \lambda_3 = \lambda_1(1 - \lambda_3)$	$-A - B + C + D + 6 = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{24}, \mathsf{p}_{56}\}$
$\tau_{11} + 2\tau_{12} + \tau_{22} - (\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1(\lambda_2 - \lambda_3) = \lambda_2(1 - \lambda_3)$	$A - B - C - D + 6 = 0$	$\{\mathsf{p}_{13}, \mathsf{p}_{26}, \mathsf{p}_{45}\}$
$4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = \lambda_2(\lambda_1 - \lambda_3)$	$-A + B - C + D + 6 = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{35}, \mathsf{p}_{26}\}$
$4\tau_{12} + \tau_{22} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_3 = \lambda_1(\lambda_2 - \lambda_3)$	$A + B + C - D + 6 = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{24}, \mathsf{p}_{35}\}$
$\tau_{11} + 4\tau_{12} + 3(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_2 - \lambda_3 = -\lambda_3(\lambda_1 - \lambda_2)$	$-A - B + C - D + 6 = 0$	$\{\mathsf{p}_{14}, \mathsf{p}_{25}, \mathsf{p}_{36}\}$
$\tau_{11} + 4\tau_{12} + \tau_{22} + 2(\tau_{11}\tau_{22} - \tau_{12}^2) = 1$	$\lambda_1 - \lambda_2 = -\lambda_1(\lambda_2 - \lambda_3)$	$A - B - C + D + 6 = 0$	$\{\mathsf{p}_{16}, \mathsf{p}_{25}, \mathsf{p}_{34}\}$

Legendre Pencil

The two-parameter twisted Legendre pencil is a double sextic surface of Picard rank 18 given by

$$\mathcal{X} : y^2 = z_0 z_1 (z_0 - z_1)(z_0 - z_2)(z_2^2 + 2\rho_1 z_1 z_2 + \rho_2^2 z_1^2), \text{ with } \rho_1, \rho_2 \in \mathbb{Q}$$

The branch locus consists of six lines, one of which is coincident with two different pairs of lines. The pencil of lines through the point $[z_0 : z_1 : z_2] = [1 : 0 : 0]$ induces an elliptic fibration on $\mathcal{X}$



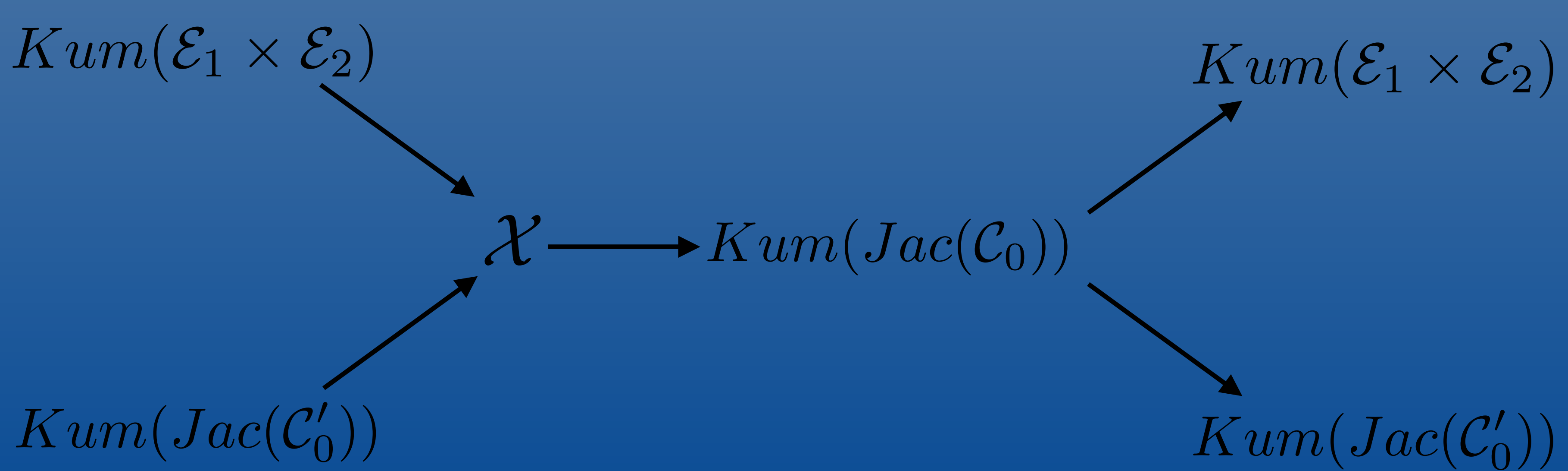
Legendre Pencil

$Kum(\mathcal{E}_1 \times \mathcal{E}_2)$ is birational to the double quadric surface obtained as a certain double cover branched along a locus of bidegree (4,4) in $\mathbb{P}^1 \times \mathbb{P}^1$. Projecting to $\mathbb{P}^1$ gives an elliptic fibration. We utilize a more complicated fibration.

The double sextic associated with $Kum(Jac(\mathcal{C}_0))$ defines an elliptic fibration. We utilize a more complicated fibration.

Choosing the even eights on $\mathcal{X}$ as the branch locus of a double cover yields new K3 surfaces. Alternatively, we quotient by a Nikulin involution produced by translation by 2-torsion section.

$$Kum(\mathcal{E}_1 \times \mathcal{E}_2) \longrightarrow \mathcal{X} \longrightarrow Kum(Jac(\mathcal{C}_0)) \longrightarrow Kum(\mathcal{E}_1 \times \mathcal{E}_2)$$



Results/Outlook

- We computed normal forms for $Kum(Jac(\mathcal{C}_0))$, $Kum(\mathcal{E}_1 \times \mathcal{E}_2)$, and $\mathcal{X}$ as quartic hyper surfaces, double quadric surfaces, double sextic surfaces, and Jacobian elliptic surfaces.
- A correspondence from a variety X to a variety Y is taken to be a subset Z of $X \times Y$ such that Z is finite and surjective over each component of X. The double coverings we constructed define correspondences between $Kum(Jac(\mathcal{C}_0))$ and $\mathcal{X}$, and $\mathcal{X}$ and $Kum(\mathcal{E}_1 \times \mathcal{E}_2)$

$$Kum(\mathcal{E}_1 \times \mathcal{E}_2) \longrightarrow \mathcal{X} \longrightarrow Kum(Jac(\mathcal{C}_0)) \longrightarrow Kum(\mathcal{E}_1 \times \mathcal{E}_2)$$

- Using the tools developed in (Malmendier, Sung, 2019) and (Braeger, Malmendier, Sung, 2020), we aim to count the number of rational points on these K3 surfaces using character sums.
- The correspondences imply isomorphisms between 4-dimensional Galois representations related to the transcendental lattice of the K3 surface X (on the one hand) and the abelian surfaces involved on the other.

Thank You!

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