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Clifford Algebras

Outline:

1. Definition of Clifford Algebra, $Cl(V,q)$
2. Universal Property of $Cl(V,q)$
3. $\mathbb{Z}_2$-graded Algebra
4. $Cl(V,q)$ and $\Lambda^* V$
5. Tensor Products
6. Periodicity

1- Clifford Algebras

Def. Let V be a vector space over the field k and suppose q is a quadratic form on V. The Clifford algebra $Cl(V,q)$ associated to V and q is an associative algebra with unit defined as follows.

Let $\mathcal{T}(V) = \bigoplus_{r=0}^{\infty} \otimes^r V$ denote the tensor algebra of V and $\mathcal{J}_q(V)$ denote the ideal in $\mathcal{T}(V)$ generated by elements of the form $v \otimes v + q(v)1$, where $v \in V$

Then, the Clifford Algebra associated w/ V and q is

$$Cl(V,q) = \mathcal{T}(V) / \mathcal{J}_q(V)$$

Remark: There is a natural embedding

$$V \hookrightarrow Cl(V,q)$$

which is the image of $V = \otimes^1 V$ under the canonical projection

$$\pi_q : \mathcal{T}(V) \longrightarrow Cl(V,q)$$

The proof of injectivity is given in "Spin Geometry".

2- Universal Property

Proposition: Let $f: V \longrightarrow \mathcal{A}$ be a linear map into an associative k-algebra with unit, such that

$$f(v) \cdot f(v) = -q(v) 1 \qquad (*)$$

for all $v \in V$. Then f extends uniquely to a k-algebra homomorphism $\tilde{f}: Cl(V,q) \longrightarrow \mathcal{A}$. Furthermore, $Cl(V,q)$ is the unique associative k-algebra with this property.

Proof: Let $f: V \to \mathcal{A}$ be a linear map. From the universal property of tensor algebras, we know that this map extends to a unique homomorphism

$$\tilde{f} : \mathcal{T}(V) \longrightarrow \mathcal{A}$$

Property $(*)$ implies that $\tilde{f} = 0$ on $\mathcal{J}_q(V)$ and $\tilde{f}$ descends to $Cl(V,q)$.

Uniqueness is left as an exercise.

Remark: Given a morphism $f:(V,q)\longrightarrow(V',q')$, i.e., a k-linear map $f:V\to V'$ between vector spaces which preserves the quadratic forms ($f^*q'=q$) there is, by our above Proposition, an induced homomorphism $\tilde{f}: Cl(V,q)\longrightarrow Cl(V',q')$.

$\boxed{\text{3- } \mathbb{Z}_2\text{-Graded Algebra}}$

The linear map on V defined by $v\mapsto -v$ preserves the quadratic form q and so by the universal property extends to an automorphism

$$\alpha: Cl(V,q)\longrightarrow Cl(V,q)$$

Since α is an involution, we can decompose $Cl(V,q)$ into positive and negative eigenspaces of α.

$$Cl(V,q) = Cl^0(V,q)\oplus Cl^1(V,q) \qquad (*)$$

where $Cl^i(V,q)=\{\varphi\in Cl(V,q) \mid \alpha(\varphi)=(-1)^i\varphi\}$.

Clearly, since $\alpha(\varphi_1\varphi_2)=\alpha(\varphi_1)\cdot\alpha(\varphi_2)$, we have that

$$Cl^i(V,q)\cdot Cl^j(V,q)\subseteq Cl^{i+j}(V,q) \qquad (**)$$

where indeces are taken modulo 2.

By definition, an algebra with a decomposition $(*)$ satisfying $(**)$ is called a $\mathbb{Z}_2$- Graded Algebra

$\boxed{\text{4- } Cl(V,q) \text{ and } \Lambda^* V}$

There is a natural filtration $\tilde{\mathcal{F}}^0\subset\tilde{\mathcal{F}}^1\subset\dots\subset \mathcal{T}(V)$ of the tensor algebra defined by

$$\tilde{\mathcal{F}}^r = \bigoplus_{s\le r}\otimes^s V$$

with the property $\tilde{\mathcal{F}}^r\otimes\tilde{\mathcal{F}}^{r'}\subseteq\tilde{\mathcal{F}}^{r+r'}$.

Through setting $\mathcal{F}^i=\pi_q(\tilde{\mathcal{F}}^i)$, we obtain a filtration $\mathcal{F}^0\subset\mathcal{F}^1\subset\dots\subset Cl(V,q)$ of our clifford algebra with the property

$$\mathcal{F}^r\otimes\mathcal{F}^{r'}\subseteq\mathcal{F}^{r+r'}$$

It follows that multiplication descends to the map

$$(\mathcal{F}^r/\mathcal{F}^{r-1})\cdot(\mathcal{F}^s/\mathcal{F}^{s-1})\longrightarrow(\mathcal{F}^{r+s}/\mathcal{F}^{r+s-1})$$

we then define the associated graded algebra

$$G^* = \bigoplus_{r\geq 0}\mathcal{F}^r/\mathcal{F}^{r-1}$$

Proposition: There is a canonical vector space isomorphism

$$\Lambda^* V\xrightarrow{\ \cong\ } Cl(V,q)$$

compatible with the filtrations.

Proof: Exercise.

See Proposition 1.2 and Proposition 1.3 of "Spin geometry".

Remark: The above map is not an isomorphism of algebras unless $q = 0$.

$$\Lambda^* V = \mathcal{J}(V) / v \otimes v$$

$\mathcal{J}_q(V)$ is generated by elements of the form $v \otimes v + q(v) 1$

5- Tensor Products

If A and B are algebras w/ unit over k, then $A \otimes B$ is the algebra whose underlying vector space is the tensor product of A and B and whose multiplication is given by

$$(a \otimes b) \cdot (a' \otimes b') = (aa') \otimes (bb')$$

If $A = A^0 \oplus A^1$ and $B = B^0 \oplus B^1$ are $\mathbb{Z}_2$-graded algebras, then we can introduce a second "$\mathbb{Z}_2$-graded" multiplication

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{\deg(b) \cdot \deg(a')} (aa') \otimes (bb') \qquad (*)$$

whenever $b \in B^i$ and $a' \in A^i$.

The resultant algebra is called the $\mathbb{Z}_2$-graded tensor algebra and is denoted by

$$A \hat{\otimes} B$$

Proposition

Let $V = V_1 \oplus V_2$ be a q-orthogonal decomposition of V (i.e. $q(v_1 + v_2) = q(v_1) + q(v_2)$, $v_1 \in V_1$, $v_2 \in V_2$). Then there is a natural isomorphism of Clifford algebras

$$Cl(V, q) \longrightarrow Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2)$$

where q_i denotes the restriction of q to V_i.

proof: Consider the map $f : V \longrightarrow Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2)$ given by

$$f(v) = v_1 \otimes 1 + 1 \otimes v_2$$

where $v = v_1 + v_2$ is the decomposition of v over our splitting $V = V_1 \oplus V_2$.

From $(*)$ and q-orthogonality,

$$f(v) \cdot f(v) = (v_1 \otimes 1 + 1 \otimes v_2)^2 = v_1^2 \otimes 1 + 1 \otimes v_2^2$$
$$= -(q_1(v_1) + q_2(v_2)) 1 \otimes 1 = -q(v) 1 \otimes 1$$

By the universal property, f extends to a unique algebra homomorphism

$$\tilde{f} : Cl(V, q) \longrightarrow Cl(V_1, q_1) \hat{\otimes} Cl(V_2, q_2)$$

Injectivity and surjectivity are left as exercises.

6- Periodicity / Examples

Of particular importance are the Clifford Algebras $Cl_{r,s} \equiv Cl(V, q)$, where $V = \mathbb{R}^{r+s}$ and

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$$

These algebras have a classical representation

Proposition: Let $e_1, \ldots, e_{r+s}$ be any q-orthonormal basis of $\mathbb{R}^{r+s} \subset Cl_{r,s}$. Then $Cl_{r,s}$ is generated (as an algebra) by $e_1, \ldots, e_{r+s}$ subject to the relations

$$e_i e_j + e_j e_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \end{cases}$$

Then $Cl_{r,s}$ is generated (as an algebra) by $e_1, \dots, e_{r+s}$

$$e_i e_j + e_j e_i = \begin{cases} -2\delta_{ij} & \text{if } i \leq r \\ +2\delta_{ij} & \text{if } i > r \end{cases}$$

Proof: Exercise.

Example: $Cl_{1,0}$ is generated by e_1 subject to the relation $e_1^2 = -1$. Let

$$\phi : Cl_{1,0} \longrightarrow \mathbb{C}$$
$$1 \longmapsto 1$$
$$e_1 \longmapsto \sqrt{-1}$$

Clearly, this is an $\mathbb{R}$-algebra isomorphism. So, as algebras over $\mathbb{R}$,

$$Cl_{1,0} \cong \mathbb{C}$$

Example: $Cl_{0,1}$ is generated by e_1 subject to the relation $e_1^2 = 1$. Let

$$\phi : Cl_{0,1} \longrightarrow \mathbb{R} \oplus \mathbb{R}$$
$$a + be_1 \longmapsto (a+b, a-b)$$

Note that

$$(a+be_1)(c+de_1) = ac + ade_1 + bce_1 + bde_1^2 = (ac+bd) + (ad+bc)e_1$$

$$\Rightarrow \phi((a+be_1)(c+de_1)) = (ac+bd+ad+bc,\ ac+bd-ad-bc) = ((a+b)(c+d),\ (a-b)(c-d)) = \phi((a+be_1)) \cdot \phi((c+de_1))$$

So, as algebras over $\mathbb{R}$,

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$$

Theorem: There are the following isomorphisms

$$Cl_{n,0} \otimes Cl_{0,2} \cong Cl_{0,n+2} \qquad (1)$$

$$Cl_{0,n} \otimes Cl_{2,0} \cong Cl_{n+2,0} \qquad (2)$$

$$Cl_{r,s} \otimes Cl_{1,1} \cong Cl_{r+1,s+1} \qquad (3)$$

for all $n, r, s \geq 0$

Proof (1): Let $e_1, \dots, e_{n+2}$ be an orthonormal basis of $\mathbb{R}^{n+2}$ in the standard inner product, and let $q(x) = -\|x\|^2$. Let $e_1', \dots, e_n'$ denote standard generators of $Cl_{n,0}$ and let e_1'', e_2'' denote standard generators of $Cl_{0,2}$.

Define a map $f : \mathbb{R}^{n+2} \longrightarrow Cl_{n,0} \otimes Cl_{0,2}$ by setting

$$f(e_i) = \begin{cases} e_i' \otimes e_1'' e_2'', & \text{for } 1 \leq i \leq n \\ 1 \otimes e_{i-n}'', & \text{for } i = n+1 \text{ or } i = n+2 \end{cases}$$

and extending linearly.

Note that for $1 \leq i, j \leq n$, we have

$$f(e_i)\cdot f(e_j) + f(e_j)\cdot f(e_i) = (e_i' e_j' + e_j' e_i') \otimes (-1) = 2\delta_{ij} 1 \otimes 1$$

and for $n+1 \leq \alpha, \beta \leq n+2$, we have

$$f(e_\alpha)\cdot f(e_\beta) + f(e_\beta)\cdot f(e_\alpha) = 1 \otimes (e_{\alpha-n}'' e_{\beta-n}'' + e_{\beta-n}'' e_{\alpha-n}'') = 2\delta_{\alpha\beta} 1 \otimes 1$$

Also, we find that $f(e_i) f(e_\alpha) + f(e_\alpha) f(e_i) = 0$. It then follows that

$$f(x)\cdot f(x) = \|x\|^2\, 1 \otimes 1, \quad \text{for all } x \in \mathbb{R}^{n+2}$$

By the Universal Property, f extends to a unique algebra homomorphism

$$\tilde{f} : Cl_{0,n+2} \longrightarrow Cl_{n,0} \otimes Cl_{0,2}$$

Injectivity and surjectivity are left as exercises.

Recall the following facts:

$$\mathbb{R}(n) \otimes \mathbb{R}(m) \cong \mathbb{R}(nm), \quad \text{for all } n, m$$

$$\cong F(n), \quad \text{for } F = \mathbb{C} \text{ or } \mathbb{H} \text{ and for all } n$$

$\mathbb{R}(n) \otimes$

$\mathbb{R}(n) \otimes_{\mathbb{R}} F \cong F(n)$, for $F = \mathbb{C}$ or $\mathbb{H}$ and for all n

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C}(2)$

$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{R}(4)$

where $F(n)$ is the $n \times n$ matrix over the field F

The complexification of $Cl_{r,s}$ is the Clifford algebra (over $\mathbb{C}$) corresponding to the complexified quadratic form

$$Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C} = Cl(\mathbb{C}^{r+s}, q \otimes \mathbb{C})$$

All non-degenerate quadratic forms on $\mathbb{C}^n$ are equivalent over $Gl_n(\mathbb{C})$. Hence, setting

$$q_{\mathbb{C}}(z) = \sum_{j=1}^{n} z_j^2$$

and defining

$$Cl_n = Cl(\mathbb{C}^n, q_{\mathbb{C}})$$

we have that

$$Cl_n \cong Cl_{n,0} \otimes_{\mathbb{R}} \mathbb{C} \cong Cl_{n-1,1} \otimes_{\mathbb{R}} \mathbb{C} \cong \dots \cong Cl_{0,n} \otimes_{\mathbb{R}} \mathbb{C} \qquad (4)$$

Example: Cl_1 is generated by e_1 subject to the relation $e_1^2 = -1$. Let

$$\phi : Cl_1 \longrightarrow \mathbb{C} \oplus \mathbb{C}$$
$$(a + b e_1) \longmapsto (a+b, a-b)$$

Showing that ϕ is a homomorphism is done in the same as we did for $Cl_{0,1}$ case. So, as algebras over $\mathbb{C}$,

$$Cl_1 \cong \mathbb{C} \oplus \mathbb{C}$$

$$\boxed{Cl_2 \cong \mathbb{C}(2)}$$

Theorem: For all $n \geq 0$, there are "periodicity" isomorphisms

$$Cl_{n+8,0} \cong Cl_{n,0} \otimes Cl_{8,0} \qquad (5)$$

$$Cl_{0,n+8} \cong Cl_{0,n} \otimes Cl_{0,8} \qquad (6)$$

$$Cl_{n+2} \cong Cl_n \otimes_{\mathbb{C}} Cl_2 \qquad (7)$$

where $Cl_{8,0} = Cl_{0,8} = \mathbb{R}(16)$ and $Cl_2 = \mathbb{C}(2)$

Proof: Exercise.